

# SOME NEW IDENTITIES ON THE $(h, q)$ -GENOCCHI NUMBERS AND POLYNOMIALS WITH WEIGHT $\alpha$

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ABSTRACT. We give some new identities for  $(h, q)$ -Genocchi numbers and polynomials by means of the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  and the weighted  $q$ -Bernstein polynomials.

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## 1. Introduction and Notations

Let  $p$  be a fixed odd prime number. Throughout this paper we use the following notations. By  $\mathbb{Z}_p$  we denote the ring of  $p$ -adic rational integers,  $\mathbb{Q}$  denotes the field of rational numbers,  $\mathbb{Q}_p$  denotes the field of  $p$ -adic rational numbers, and  $\mathbb{C}_p$  denotes the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ . The  $p$ -adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$

In this paper, we assume  $|q - 1|_p < 1$  as an indeterminate. Let  $UD(\mathbb{Z}_p)$  be the space of uniformly differentiable functions on  $\mathbb{Z}_p$ . For  $f \in UD(\mathbb{Z}_p)$ , the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  is defined by T. Kim:

$$(1.1) \quad I_{-q}(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi) = \lim_{N \rightarrow \infty} \sum_{\xi=0}^{p^N-1} q^\xi f(\xi) (-1)^\xi$$

(for more informations on this subject, see [29], [30] and [31]).

From (1.1), we have well known the following equality:

$$(1.2) \quad qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0)$$

here  $f_1(x) := f(x+1)$  (for details, see [2-40]).

Let  $C([0, 1])$  be the space of continuous functions on  $[0, 1]$ . For  $C([0, 1])$ , the weighted  $q$ -Bernstein operator for  $f$  is defined by

$$\mathcal{B}_{n,q}^{(\alpha)}(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}^{(\alpha)}(x | q) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}$$

where  $n, k \in \mathbb{N}^*$ . Here  $B_{k,n}^{(\alpha)}(x | q)$  is called weighted  $q$ -Bernstein polynomials, which are defined by

$$(1.3) \quad B_{k,n}^{(\alpha)}(x | q) = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}, \quad x \in [0, 1]$$

(for more informations on this subject, see [3], [33], [39] and [40]).

As is well known, the ordinary Genocchi polynomials are defined by means of the following generating function:

$$(1.4) \quad \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = e^{G(x)t} = \frac{2t}{e^t + 1} e^{xt}.$$

where the usual convention about replacing  $G^n(x)$  by  $G_n(x)$ . For  $x = 0$  in (1.4), we have to  $G_n(0) := G_n$ , which is called Genocchi numbers. Then, we can write the following

$$(1.5) \quad e^{Gt} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t + 1}.$$

In [4], the  $q$ -Genocchi numbers are given as

$$G_{0,q} = 0 \text{ and } q(qG_q + 1)^n + G_{n,q} = \begin{cases} [2]_q & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

where the usual convention about replacing  $(G_q)^n$  by  $G_{n,q}$ .

For any  $n \in \mathbb{N}^*$ , the  $(h, q)$ -Genocchi numbers are defined by

$$G_{0,q}^{(h)} = 0 \text{ and } q^{h-1} (qG_q^{(h)} + 1)^n + G_{n,q}^{(h)} = \begin{cases} [2]_q & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

where the usual convention about replacing  $(G_q^{(h)})^n$  by  $G_{n,q}^{(h)}$  (for details, see [11]).

Recently, Araci *et al.* are defined the  $(h, q)$ -Genocchi numbers with weight  $\alpha$  by

$$(1.6) \quad \frac{\tilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1} = \int_{\mathbb{Z}_p} q^{(h-1)\xi} [x + \xi]_{q^\alpha}^n d\mu_{-q}(\xi).$$

By (1.6), we have the following identity

$$(1.7) \quad \tilde{G}_{n,q}^{(\alpha,h)}(x) = \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} \tilde{G}_{n,q}^{(\alpha,h)} [x]_{q^\alpha}^{n-k} = q^{-\alpha x} \left( q^{\alpha x} \tilde{G}_q^{(\alpha,h)} + [x]_{q^\alpha} \right)^n$$

where the usual convention about replacing  $(\tilde{G}_q^{(\alpha,h)})^n$  by  $\tilde{G}_{n,q}^{(\alpha,h)}$  is used (for details, [5]).

In this paper, we derive some new properties  $(h, q)$ -Genocchi numbers and polynomials from the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ . Also, we show that these type polynomials are related to  $(h, q)$ -Genocchi numbers and polynomials.

## 2. On the $(h, q)$ -Genocchi numbers and polynomials

In this section, we consider the  $(h, q)$ -Genocchi numbers and polynomials by using fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  and the weighted  $q$ -Bernstein polynomials. We can now start the following expression.

In [5], we have the  $(h, q)$ -Genocchi numbers as follows: For  $\alpha \in \mathbb{N}^*$  and  $n, h \in \mathbb{N}$ ,

$$(2.1) \quad \tilde{G}_{0,q}^{(\alpha,h)} = 0 \text{ and } q^h \tilde{G}_{n,q}^{(\alpha,h)}(1) + \tilde{G}_{n,q}^{(\alpha,h)} = \begin{cases} [2]_q & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

By (1.7) and (2.1), we have the following corollary.

**Corollary 1.** For  $\alpha \in \mathbb{N}^*$  and  $n, h \in \mathbb{N}$ , then we have

$$(2.2) \quad \tilde{G}_{0,q}^{(\alpha,h)} = 0 \text{ and } q^{h-\alpha} \left( q^\alpha \tilde{G}_q^{(\alpha,h)} + 1 \right)^n + \tilde{G}_{n,q}^{(\alpha,h)} = \begin{cases} [2]_q & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

By (1.6), we get symmetric property that

$$\begin{aligned} \frac{\tilde{G}_{n+1,q^{-1}}^{(\alpha,h)}(1-x)}{n+1} &= \int_{\mathbb{Z}_p} q^{(1-h)\xi} [1-x+\xi]_{q^{-\alpha}}^n d\mu_{-q^{-1}}(\xi) \\ &= (-1)^n q^{h+\alpha n-1} \int_{\mathbb{Z}_p} q^{(h-1)\xi} [x+\xi]_{q^\alpha}^n d\mu_{-q}(\xi) \end{aligned}$$

From this, we state the following theorem.

**Theorem 1.** The following identity

$$(2.3) \quad \tilde{G}_{n+1,q^{-1}}^{(\alpha,h)}(1-x) = (-1)^n q^{h+\alpha n-1} \tilde{G}_{n+1,q}^{(\alpha,h)}(x)$$

is true.

By using (1.7), (2.1) and (2.2), we compute as follows:

$$\begin{aligned} (2.4) \quad q^{2\alpha} \tilde{G}_{n,q}^{(\alpha,h)}(2) &= \left( q^{2\alpha} \tilde{G}_q^{(\alpha,h)} + [2]_{q^\alpha} \right)^n \\ &= \sum_{l=0}^n \binom{n}{l} q^{\alpha l} \left( q^\alpha \tilde{G}_q^{(\alpha,h)} + 1 \right)^l \\ &= nq^{2\alpha-h} \left( [2]_q - \tilde{G}_{1,q}^{(\alpha,h)} \right) - q^{\alpha-h} \sum_{l=2}^n \binom{n}{l} q^{\alpha l} \tilde{G}_{l,q}^{(\alpha,h)} \\ &= nq^{2\alpha-h} [2]_q + q^{2\alpha-2h} \tilde{G}_{n,q}^{(\alpha,h)} \text{ if } n > 1. \end{aligned}$$

After the above applications, we procure the following theorem.

**Theorem 2.** For  $n > 1$ , then we have

$$\tilde{G}_{n,q}^{(\alpha,h)}(2) = nq^{-h} [2]_q + q^{-2h} \tilde{G}_{n,q}^{(\alpha,h)}.$$

We need the following equality for sequel of this paper:

$$(2.5) \quad [1-x]_{q^{-\alpha}}^n = \left( \frac{1-q^{-\alpha(1-x)}}{1-q^{-\alpha}} \right)^n = (-1)^n q^{n\alpha} [x-1]_{q^\alpha}^n.$$

Now also, by using (2.5), we consider the following

$$\begin{aligned} &q^{h-1} \int_{\mathbb{Z}_p} q^{(h-1)\xi} [1-\xi]_{q^{-\alpha}}^n d\mu_{-q}(\xi) \\ &= (-1)^n q^{h+n\alpha-1} \int_{\mathbb{Z}_p} q^{(h-1)\xi} [\xi-1]_{q^\alpha}^n d\mu_{-q}(\xi) \\ &= (-1)^n q^{h+n\alpha-1} \frac{\tilde{G}_{n+1,q}^{(\alpha,h)}(-1)}{n+1}. \end{aligned}$$

By considering last identity and (2.3), we get the following theorem.

**Theorem 3.** The following identity holds true:

$$(2.6) \quad \int_{\mathbb{Z}_p} q^{(h-1)(\xi+1)} [1-\xi]_{q^{-\alpha}}^n d\mu_{-q}(\xi) = \frac{\tilde{G}_{n+1,q^{-1}}^{(\alpha,h)}(2)}{n+1}.$$

From (2.6), we have the following

$$\int_{\mathbb{Z}_p} q^{(h-1)\xi} [1 - \xi]_{q^{-\alpha}}^n d\mu_{-q}(\xi) = [2]_q + q^{h+1} \frac{\tilde{G}_{n+1, q^{-1}}^{(\alpha, h)}}{n+1}.$$

Thus, we obtain the following theorem.

**Theorem 4.** *The following identity*

$$(2.7) \quad \int_{\mathbb{Z}_p} q^{(h-1)\xi} [1 - \xi]_{q^{-\alpha}}^n d\mu_{-q}(\xi) = [2]_q + q^{h+1} \frac{\tilde{G}_{n+1, q^{-1}}^{(\alpha, h)}}{n+1}$$

is true.

### 3. Some new identities on the $(h, q)$ -Genocchi numbers

In this section, we introduce new identities of the  $(h, q)$ -Genocchi numbers, that is, we derive some interesting and worthwhile relations for studying in Theory of Analytic Numbers.

For  $x \in [0, 1]$ , we give definition of weighted  $q$ -Bernstein polynomials as follows:

$$(3.1) \quad B_{k,n}^{(\alpha)}(x | q) = \binom{n}{k} [x]_{q^\alpha}^k [1 - x]_{q^{-\alpha}}^{n-k}, \text{ where } n, k \in \mathbb{Z}_+.$$

By expression of (3.1), we have the properties of symmetry of weighted  $q$ -Bernstein polynomials as follows:

$$(3.2) \quad B_{k,n}^{(\alpha)}(x | q) = B_{n-k,n}^{(\alpha)}\left(1 - x \mid \frac{1}{q}\right), \text{ (for details, see [33]).}$$

Thus, (2.7), (3.1) and (3.2), we see that

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n}^{(\alpha)}(x | q) d\mu_{-q}(x) = \binom{n}{k} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q^\alpha}^k [1 - x]_{q^{-\alpha}}^{n-k} d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} q^{(h-1)x} [1 - x]_{q^{-\alpha}}^{n-l} d\mu_{-q}(x) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \left\{ [2]_q + q^{h+1} \frac{\tilde{G}_{n-l+1, q^{-1}}^{(\alpha, h)}}{n-l+1} \right\} \\ &= \begin{cases} [2]_q + q^{h+1} \frac{\tilde{G}_{n+1, q^{-1}}^{(\alpha, h)}}{n+1} & \text{if } k = 0, \\ \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \left\{ [2]_q + q^{h+1} \frac{\tilde{G}_{n-l+1, q^{-1}}^{(\alpha, h)}}{n-l+1} \right\} & \text{if } k \neq 0. \end{cases} \end{aligned}$$

On the other hand, for  $n, k \in \mathbb{Z}_+$  with  $n > k$ , we compute

$$\begin{aligned}
I_2 &= \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n}^{(\alpha)}(x | q) d\mu_{-q}(x) \\
&= \binom{n}{k} \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q^\alpha}^{l+k} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{\tilde{G}_{l+k+1,q}^{(\alpha,h)}}{l+k+1}.
\end{aligned}$$

Equating  $I_1$  and  $I_2$ , then we have the following theorem.

**Theorem 5.** *The following identity holds true:*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{\tilde{G}_{l+k+1,q}^{(\alpha,h)}}{l+k+1} = \begin{cases} [2]_q + q^{h+1} \frac{\tilde{G}_{n+1,q^{-1}}^{(\alpha,h)}}{n+1} & \text{if } k = 0, \\ \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \left\{ [2]_q + q^{h+1} \frac{\tilde{G}_{n-l+1,q^{-1}}^{(\alpha,h)}}{n-l+1} \right\} & \text{if } k \neq 0. \end{cases}$$

Let  $n_1, n_2, k \in \mathbb{Z}_+$  with  $n_1 + n_2 > 2k$ . Then, we derive the followings

$$\begin{aligned}
I_3 &= \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n_1}^{(\alpha)}(x | q) B_{k,n_2}^{(\alpha)}(x | q) d\mu_{-q}(x) \\
&= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}_p} q^{(h-1)x} [1-x]_{q^{-\alpha}}^{n_1+n_2-l} d\mu_{-q}(x) \\
&= \left( \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left( [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2-l+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2-l+1} \right) \right) \\
&= \begin{cases} [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2+1,q^{-1}}^{(\alpha,h)}}{n+1} & \text{if } k = 0, \\ \binom{n}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left\{ [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2-l+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2-l+1} \right\} & \text{if } k \neq 0. \end{cases}
\end{aligned}$$

In other words, by using the binomial theorem, we can derive the following equation.

$$\begin{aligned}
I_4 &= \int_{\mathbb{Z}_p} q^{(h-1)x} B_{k,n_1}^{(\alpha)}(x | q) B_{k,n_2}^{(\alpha)}(x | q) d\mu_{-q}(x) \\
&= \prod_{i=1}^2 \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \int_{\mathbb{Z}_p} q^{(h-1)x} [x]_{q^\alpha}^{2k+l} d\mu_{-q}(x) \\
&= \prod_{i=1}^2 \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \frac{\tilde{G}_{l+2k+1,q}^{(\alpha,h)}}{l+2k+1}.
\end{aligned}$$

Combining  $I_3$  and  $I_4$ , we state the following theorem.

**Theorem 6.** For  $n_1, n_2, k \in \mathbb{Z}_+$  with  $n_1 + n_2 > 2k$ , we have

$$\begin{aligned} & \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \frac{\tilde{G}_{l+2k+1,q}^{(\alpha,h)}}{l+2k+1} \\ &= \begin{cases} [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2+1} & \text{if } k = 0, \\ \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left\{ [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2-l+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2-l+1} \right\} & \text{if } k \neq 0. \end{cases} \end{aligned}$$

For  $x \in \mathbb{Z}_p$  and  $s \in \mathbb{N}$  with  $s \geq 2$ , let  $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$  with  $\sum_{l=1}^s n_l > sk$ . Then we take the fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  for the weighted  $q$ -Bernstein polynomials of degree  $n$  as follows:

$$\begin{aligned} I_5 &= \int_{\mathbb{Z}_p} q^{(h-1)x} \left\{ \prod_{i=1}^s B_{k,n_i}^{(\alpha)}(x | q) \right\} d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \int_{\mathbb{Z}_p} [x]_{q^\alpha}^{sk} [1-x]_{q^{-\alpha}}^{n_1+n_2+\dots+n_s-sk} q^{(h-1)x} d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n_1+n_2+\dots+n_s-l} q^{(h-1)x} d\mu_{-q}(x) \\ &= \begin{cases} [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2+\dots+n_s+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2+\dots+n_s+1} & \text{if } k = 0, \\ \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2+\dots+n_s-l+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2+\dots+n_s-l+1} \right\} & \text{if } k \neq 0. \end{cases} \end{aligned}$$

On the other hand, from the definition of weighted  $q$ -Bernstein polynomials and the binomial theorem, we easily get

$$\begin{aligned} I_6 &= \int_{\mathbb{Z}_p} q^{(h-1)x} \left\{ \prod_{i=1}^s B_{k,n_i}^{(\alpha)}(x | q) \right\} d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^s (n_d - k)}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_{q^\alpha}^{sk+l} q^{(h-1)x} d\mu_{-q}(x) \\ &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^s (n_d - k)}{l} (-1)^l \frac{\tilde{G}_{l+sk+1,q}^{(\alpha,h)}}{l+sk+1}. \end{aligned}$$

Equating  $I_5$  and  $I_6$ , we discover the following theorem.

**Theorem 7.** For  $s \in \mathbb{N}$  with  $s \geq 2$ , let  $n_1, n_2, \dots, n_s, k \in \mathbb{Z}_+$  with  $\sum_{l=1}^s n_l > sk$ . Then, we have

$$\begin{aligned} & \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^s (n_d - k)}{l} (-1)^l \frac{\tilde{G}_{l+sk+1,q}^{(\alpha,h)}}{l+sk+1} \\ &= \begin{cases} [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2+\dots+n_s+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2+\dots+n_s+1} & \text{if } k = 0, \\ \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left\{ [2]_q + q^{h+1} \frac{\tilde{G}_{n_1+n_2+\dots+n_s-l+1,q^{-1}}^{(\alpha,h)}}{n_1+n_2+\dots+n_s-l+1} \right\} & \text{if } k \neq 0. \end{cases} \end{aligned}$$

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